

Coherent states for quadratic Hamiltonians

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Abstract

The coherent states for a set of quadratic Hamiltonians in the trap regime are constructed. A matrix technique which allows to identify directly the creation and annihilation operators will be presented. Then, the coherent states as simultaneous eigenstates of the annihilation operators will be derived, and they are going to be compared with those attained through the displacement operator method. The corresponding wave function will be found, and a general procedure for obtaining several mean values involving the canonical operators in these states will be described. The results will be illustrated through the asymmetric Penning trap.

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1 Introduction

As it was shown by Schrödinger in 1926 for the harmonic oscillator, the quasi-classical states are important for the description of physical systems in the classical limit (see e.g. [1]). The catchy term *coherent states* (CS) was used for the first time by Glauber long after, when studying electromagnetic correlation functions [2,3]. With this application it was realized that the CS are useful as well in the intrinsically quantum domain. Indeed, the CS approach is nowadays widely employed for dealing with quantum physical systems. According to Glauber there are three equivalent ways to construct the CS for the harmonic oscillator. The first one is to define them as eigenstates of the annihilation operator. The second one is to build the CS through the action of a displacement operator onto the ground state. The third way is to consider them as quantum states having a minimum Heisenberg uncertainty relationship. These three properties can be used as definitions to build the CS for systems different from the harmonic oscillator. However, it is noteworthy that each of them leads to sets of CS which do not coincide in general [4–13]. In fact, even for the harmonic oscillator the third definition does not produce just the standard CS, since it includes as well the so-called squeezed states [14, 15].

In spite of its long term life, from time to time there are some advances which maintain the subject alive. This is the case, e.g., of the recently discovered coherent states for a charged particle inside an ideal Penning trap [16]. Since the corresponding Hamiltonian is quadratic in the position and momentum operators, one would expect that the CS appear as a generalized displacement operator acting onto the corresponding ground state. However, it is worth to notice that the Penning trap Hamiltonian does not have any ground state at all, since it is not a positively

defined operator. Despite, it was possible to implement in a simple way the corresponding CS construction. Thus, we need to take into account this Hamiltonian property when studying the coherent states of general systems.

In this article we are going to address the CS construction for systems characterized by a certain set of quadratic Hamiltonians. The CS will be built up as simultaneous eigenstates of the corresponding annihilation operators, and also by applying a generalized displacement operator onto an appropriate extremal state. We will see that in the case of a positively defined Hamiltonian this extremal state will coincide with the ground state. In order to perform the CS construction, we need to find first the annihilation and creation operators. This task will be done by using a matrix technique, which generalizes the one employed in [16] (see also [17–20]). In this way, we will simply and systematically identify the characteristic algebra of the involved Hamiltonians. Our procedure represents a generalization to dimensions greater than one of the standard technique to deal with the harmonic oscillator, which is closely related to the well known factorization method (see, e.g., [21, 22]).

The paper is organized as follows. In section 2 we will introduce a detailed recipe for systematically obtaining the annihilation and creation operators for quadratic Hamiltonians in the trap regime. The coherent states derivation shall be elaborated in section 3, while in section 4 we will address the completeness of this set of CS, we shall obtain the mean values of several important physical quantities and the time evolution of these states. We are going to apply our general results to an asymmetric Penning trap in section 5, and our conclusions will be presented at section 6.

2 Ladders operators for quadratic Hamiltonians

Along this work we are going to consider a general set of n -dimensional quadratic Hamiltonians of the form

$$H = \frac{1}{2}\eta^T \mathbf{B} \eta, \quad (1)$$

where \mathbf{B} is a $2n \times 2n$ real constant symmetric matrix, $\eta = (\vec{X}, \vec{P})^T$, and \vec{X}, \vec{P} are the n -dimensional coordinate and momentum operators in the Schrödinger picture satisfying the canonical commutation relationships $[X_i, P_j] = i\delta_{ij}$ (notice that a system of units such that $\hbar = 1$ will be used throughout this paper). The time evolution of the operator vector $\eta(t) = U^\dagger(t)\eta U(t)$ in the Heisenberg picture is governed by

$$\frac{d\eta(t)}{dt} = U^\dagger(t)[iH, \eta]U(t) = U^\dagger(t)\mathbf{\Lambda}\eta U(t) = \mathbf{\Lambda}\eta(t), \quad (2)$$

$U(t)$ being the evolution operator of the system such that $U(0) = 1$, and $\mathbf{\Lambda} = \mathbf{J}\mathbf{B}$, where \mathbf{J} is the well known $2n \times 2n$ matrix

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}, \quad (3)$$

satisfying

$$\mathbf{J}^T = -\mathbf{J}, \quad \mathbf{J}^2 = -\mathbf{1}_{2n}, \quad \det(\mathbf{J}) = 1, \quad (4)$$

in which $\mathbf{1}_m$ represents the $m \times m$ identity matrix. The solution of Eq.(2) is given by

$$\eta(t) = e^{\mathbf{\Lambda}t}\eta(0) = e^{\mathbf{\Lambda}t}\eta. \quad (5)$$

In order to identify the annihilation and creation operators of H , we need to find the right and left eigenvectors of Λ . Since in general Λ is non-hermitian, its right and left eigenvectors are not necessarily adjoint to each other.

Let us consider in the first place the $2n$ -th order characteristic polynomial of Λ , $P(\lambda) = \det(\Lambda - \lambda) = \det(\mathbf{JB} - \lambda)$. Using Eqs.(4) and the fact that \mathbf{B} is a symmetric matrix, we obtain

$$P(\lambda) = \det(\mathbf{JB} - \lambda) = \det[(\mathbf{JB} - \lambda)^T] = \det(\mathbf{JB} + \lambda) = P(-\lambda). \quad (6)$$

This means that if λ is an eigenvalue of Λ , then $-\lambda$ also will be. Throughout this work we are going to denote the eigenvalues of Λ as λ_k and $-\lambda_k$, taking λ_k in the way

$$\text{Re}(\lambda_k) > 0 \quad \text{or} \quad \text{Im}(\lambda_k) > 0 \text{ if } \text{Re}(\lambda_k) = 0, \quad k = 1, \dots, n. \quad (7)$$

Let us label as u_k^\pm and f_k^\pm the right and left eigenvectors associated to the eigenvalues $\pm\lambda_k$ respectively, i.e.,

$$\Lambda u_k^\pm = \pm\lambda_k u_k^\pm, \quad f_k^\pm \Lambda = \pm\lambda_k f_k^\pm. \quad (8)$$

Notice that both u_k^\pm and f_k^\pm can be determined from Eq.(8) up to arbitrary factors. Part of this arbitrariness will be eliminated by imposing two requirements. The first one is that the right and left eigenvectors be dual to each other, namely,

$$f_j^r u_k^{r'} = \delta_{jk} \delta_{rr'}, \quad (9)$$

with $j, k = 1, \dots, n$ and $r, r' = +, -$. The second condition, which is needed in order to recover the standard annihilation and creation operators for the one-dimensional harmonic oscillator, is to ask that the left eigenvectors f_k^\pm involved in the commutators

$$[f_k^- \eta, f_k^+ \eta] = \gamma_k, \quad \gamma_k \in \mathbb{C}, \quad (10)$$

are such that

$$|\gamma_k| = 1. \quad (11)$$

In this paper we are going to discuss just the case in which there are no degeneracies in the eigenvalues $\pm\lambda_k$ so that the identity matrix $\mathbf{1}_{2n}$ can be expanded as (see [23])

$$\mathbf{1}_{2n} = \sum_{k=1}^n (u_k^+ \otimes f_k^+ + u_k^- \otimes f_k^-), \quad (12)$$

with \otimes representing tensor product. Then we get

$$\begin{aligned} \eta(t) &= e^{\Lambda t} \left[\sum_{k=1}^n (u_k^+ \otimes f_k^+ + u_k^- \otimes f_k^-) \right] \eta = \sum_{k=1}^n [e^{\lambda_k t} u_k^+ \otimes f_k^+ \eta + e^{-\lambda_k t} u_k^- \otimes f_k^- \eta] \\ &= \sum_{k=1}^n (e^{\lambda_k t} u_k^+ L_k^+ + e^{-\lambda_k t} u_k^- L_k^-), \end{aligned} \quad (13)$$

where $L_k^\pm \equiv f_k^\pm \eta$.

It is worth to point out that, in the classical case, L_k^\pm represent c -numbers which are related with the initial conditions. The time dependence of $\eta(t)$ is determined essentially by the λ_k -values, which are complex in general. If $\text{Re}(\lambda_k) \neq 0$ it can be seen that one of the two involved exponentials of the k -th term in the previous relation diverges as t increases and, thus, the classical motion will be in general unbounded (see [24–27]). The only way in which this does not happen is that all the eigenvalues be purely imaginary so that the corresponding exponentials will just induce oscillations in time, and therefore in this case the classical evolution of the vector $\eta(t)$ will remain always bounded.

On the other hand, in the quantum regime the L_k^\pm are linear operators in the canonical variables \vec{X}, \vec{P} . It is straightforward to show that their commutators with H reduce to

$$[H, L_k^\pm] = \mp i\lambda_k L_k^\pm. \quad (14)$$

In addition, it turns out that

$$[L_j^-, L_k^-] = [L_j^+, L_k^+] = 0, \quad [L_j^-, L_k^+] = 0, \quad k \neq j. \quad (15)$$

However,

$$[L_k^-, L_k^+] = \gamma_k \neq 0, \quad k = 1, \dots, n. \quad (16)$$

Eq.(14) implies that L_k^\pm behave, at least formally, as ladders operators for the eigenvectors of H , changing its eigenvalues by $\mp i\lambda_k$. However, this statement has to be managed carefully since it could happen that the action of L_k^\pm onto an eigenvector of H produces something which does not belong to the domain of H , and in this case we would not get new eigenvectors of H . In the next section we will explore an interesting situation (the trap regime of our systems) for which the application of L_k^\pm onto an eigenvector of H produce a new one with different eigenvalue.

Let us point out that Eqs.(14-16) imply that H can be expressed in a simple way in terms of $\{L_k^\pm, k = 1, \dots, n\}$ [24]. This is a consequence of the following general theorem:

Theorem. If \mathcal{L} is an irreducible algebra of operators generated by L_i^\pm which obey $[L_i^+, L_j^+] = [L_i^-, L_j^-] = 0$, $[L_i^-, L_j^+] = \gamma_i \delta_{ij}$ with $|\gamma_i| = 1$, $i, j = 1, \dots, n$, then an operator $H \in \mathcal{L}$ which fulfills relation (14) can be written as

$$H = \sum_{k=1}^n \left(\frac{-i\lambda_k}{\gamma_k} \right) L_k^+ L_k^- + g_0, \quad (17)$$

where $g_0 \in \mathbb{C}$.

Demonstration. Actually, due to Eqs.(14-16) it turns out that

$$g_0 \equiv H - \sum_{k=1}^n \left(\frac{-i\lambda_k}{\gamma_k} \right) L_k^+ L_k^-$$

commutes with L_k^\pm for all k , thus with any function of them, and so g_0 must be a c -number. \square

From now on we are going to discard situations such that $\text{Re}(\lambda_k) \neq 0$ for some $k = 1, \dots, n$, restricting ourselves to cases in which the λ_k are purely imaginary for all k , i.e., we stick to the trap regime of our systems.

2.1 Algebraic structure of H in the trap regime

Let us suppose that $\lambda_k = i\omega_k$ with $\omega_k > 0$, $k = 1, \dots, n$. Hence, $\lambda_k^* = -\lambda_k$, and since Λ is real, without loosing generality we can choose

$$f_k^- = (f_k^+)^*, \quad u_k^- = (u_k^+)^*. \quad (18)$$

Since L_k^\pm are linear combinations of the hermitian components of $\eta(X_1, \dots, X_n, P_1, \dots, P_n)$, it turns out that

$$(L_k^\pm)^\dagger = L_k^\mp. \quad (19)$$

Note moreover that $\gamma_k^* = \gamma_k$, i.e., $\gamma_k \in \mathbb{R}$, and the use of Eq.(11) implies that $\gamma_k = \pm 1$. Summarizing these results, Eq.(14) becomes in this case

$$[H, L_k^\pm] = \pm \omega_k L_k^\pm, \quad (20)$$

i.e., the modifications suffered by the eigenvalues of H through the action of the ladder operators L_k^\pm are given by the real quantities $\pm \omega_k$. In addition, H is factorized as (see Eq.(17))

$$H = \sum_{k=1}^n \gamma_k \omega_k L_k^+ L_k^- + g_0, \quad g_0 \in \mathbb{R}. \quad (21)$$

Notice that the previous summation either involves terms for which $\gamma_k = 1$, which are of the oscillator kind since they are positively defined, or terms with $\gamma_k = -1$ which are of the *anti-oscillator* type since they are negatively defined. Thus, it is natural to define a *global algebraic structure* for our system (see, e.g., [16, 28]), which is independent of the spectral details but has to do with the fact that any quadratic Hamiltonian in the trap regime can be expressed in terms of several independent oscillators, some of them indeed being anti-oscillators (compare Eq.(21)). This global structure is characterized mathematically by identifying n sets of number, annihilation and creation operators of the system, $\{N_k, B_k, B_k^\dagger\}$, $k = 1, \dots, n$, in the way:

$$\begin{aligned} B_k &= L_k^-, & B_k^\dagger &= L_k^+, & \text{for } \gamma_k &= 1, \\ B_k &= L_k^+, & B_k^\dagger &= L_k^-, & \text{for } \gamma_k &= -1, \\ N_k &= B_k^\dagger B_k, & k &= 1, \dots, n, \end{aligned} \quad (22)$$

so that the standard commutation relationships are satisfied,

$$\begin{aligned} [B_j, B_k^\dagger] &= \delta_{jk}, & [B_j, B_k] &= [B_j^\dagger, B_k^\dagger] = 0, \\ [N_k, B_k] &= -B_k, & [N_k, B_k^\dagger] &= B_k^\dagger, & j, k &= 1, \dots, n. \end{aligned} \quad (23)$$

Let us construct now a basis $\{|n_1, \dots, n_n\rangle, n_j = 0, 1, 2, \dots, j = 1, 2, \dots, n\}$ of common eigenstates of $\{N_1, \dots, N_n\}$ (the Fock states)

$$N_j |n_1, \dots, n_n\rangle = n_j |n_1, \dots, n_n\rangle, \quad j = 1, \dots, n, \quad (24)$$

starting from an *extremal state* $|0, \dots, 0\rangle$, which is annihilated simultaneously by B_1, \dots, B_n :

$$B_j |0, \dots, 0\rangle = 0, \quad j = 1, \dots, n. \quad (25)$$

If we assume that $|0, \dots, 0\rangle$ is normalized, it turns out that:

$$|n_1, \dots, n_n\rangle = \frac{B_1^{\dagger n_1} \dots B_n^{\dagger n_n} |0, \dots, 0\rangle}{\sqrt{n_1! \dots n_n!}}. \quad (26)$$

Moreover, B_j and B_j^\dagger , $j = 1, \dots, n$, act onto $|n_1, \dots, n_n\rangle$ in a standard way:

$$\begin{aligned} B_j |n_1, \dots, n_{j-1}, n_j, n_{j+1}, \dots, n_n\rangle &= \sqrt{n_j} |n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_n\rangle, \\ B_j^\dagger |n_1, \dots, n_{j-1}, n_j, n_{j+1}, \dots, n_n\rangle &= \sqrt{n_j + 1} |n_1, \dots, n_{j-1}, n_j + 1, n_{j+1}, \dots, n_n\rangle. \end{aligned} \quad (27)$$

Now, in terms of the operators $\{B_j, B_j^\dagger, j = 1, \dots, n\}$ our Hamiltonian is expressed by

$$H = \sum_{k=1}^n \gamma_k \omega_k B_k^\dagger B_k + g'_0. \quad (28)$$

It is clear that the Fock states $|n_1, \dots, n_n\rangle$ are eigenstates of H with eigenvalues $E_{n_1, \dots, n_n} = \gamma_1 \omega_1 n_1 + \dots + \gamma_n \omega_n n_n + g'_0 \equiv E(n_1, \dots, n_n)$. In particular, the extremal state $|0, \dots, 0\rangle$ has eigenvalue $E_{0, \dots, 0} = g'_0$. In case that $\gamma_k = 1$ for all k , then $H - g'_0$ will be a positively defined operator, and the extremal state $|0, \dots, 0\rangle$ will become the ground state for our system, associated to the lowest eigenvalue $E_{0, \dots, 0} = g'_0$ of H . On the other hand, if there is at least one index j for which $\gamma_j = -1$, then $H - g'_0$ will not be positively defined, since the corresponding j -th term is of *inverted oscillator* type, and the state $|0, \dots, 0\rangle$ will not be a ground state for our system (however it keeps its extremal nature since it is always annihilated by the n operators $B_j, j = 1, \dots, n$).

Following [16, 28] it is straightforward to see that, besides the global algebraic structure, there is an *intrinsic algebraic structure* for our system, characterized by the existing relationship between the Hamiltonian H and the n number operators N_k :

$$H = E(N_1, \dots, N_n) = \sum_{k=1}^n \gamma_k \omega_k N_k + g'_0. \quad (29)$$

As in the examples discussed in [16, 28], it turns out that this intrinsic algebraic structure is responsible for the specific spectrum of our Hamiltonian. On the other hand, the global algebraic structure arises from the existence of the n independent oscillator modes for H , each one characterized by the standard generators $\{N_j, B_j, B_j^\dagger\}$, $j = 1, \dots, n$. This global behavior allows us to identify in a natural way the extremal state $|0, \dots, 0\rangle \equiv |\mathbf{0}\rangle$, which plays the role of a ground state although it does not necessarily has a minimum energy eigenvalue. Moreover, the very existence of the extremal state $|\mathbf{0}\rangle$ is guaranteed by a theorem [24] ensuring that if the operators $\{B_1, \dots, B_n\}$ obey the commutation relations given by Eq.(23), then the system of partial differential equations

$$\langle \vec{x} | B_j | 0, \dots, 0 \rangle = \langle \vec{x} | B_j | \mathbf{0} \rangle = 0, \quad j = 1, \dots, n, \quad (30)$$

has the square integrable solution

$$\phi_{\mathbf{0}}(\vec{x}) = \langle \vec{x} | \mathbf{0} \rangle = c e^{-\frac{1}{2} a_{ij} x_i x_j} = c e^{-\frac{1}{2} (\vec{x}^T \mathbf{a} \vec{x})}, \quad (31)$$

with c being a normalization factor. In this wave function, $\mathbf{a} = (a_{ij})$ represents a symmetric matrix whose complex entries are found by solving the system of equations (30), leading to

$$\mathbf{a}\vec{\alpha}_j = \vec{\beta}_j, \quad j = 1, \dots, n, \quad (32)$$

where $\vec{\alpha}_j$ and $\vec{\beta}_j$ are obtained by expressing B_j and B_j^\dagger as

$$B_j = i\vec{P} \cdot \vec{\alpha}_j + \vec{X} \cdot \vec{\beta}_j, \quad B_j^\dagger = -i\vec{\alpha}_j^\dagger \cdot \vec{P} + \vec{\beta}_j^\dagger \cdot \vec{X}, \quad j = 1, \dots, n. \quad (33)$$

The wave functions for the other Fock states can be found from Eq.(26).

3 Coherent States

Once our Hamiltonian has been expressed appropriately in terms of annihilation and creation operators, we can develop a similar treatment as for the harmonic oscillator to build up the corresponding coherent states. Here we are going to construct them either as simultaneous eigenstates of the annihilation operators of the system or as the ones resulting from acting the global displacement operator onto the extremal state.

3.1 Annihilation Operator Coherent States (AOCS)

In the first place let us look for the annihilation operator coherent states (AOCS) as common eigenstates of the B_j 's:

$$B_j|z_1, \dots, z_n\rangle = z_j|z_1, \dots, z_n\rangle, \quad z_j \in \mathbb{C}, \quad j = 1, \dots, n. \quad (34)$$

Following a standard procedure, let us expand them in the basis $\{|n_1, \dots, n_n\rangle\}$:

$$|z_1, \dots, z_n\rangle = \sum_{n_1, \dots, n_n=0}^{\infty} c_{n_1, \dots, n_n} |n_1, \dots, n_n\rangle. \quad (35)$$

By imposing now that Eq.(34) is satisfied, the following recurrence relationships are obtained,

$$c_{n_1, \dots, n_j, \dots, n_n} = \frac{z_j}{\sqrt{n_j}} c_{n_1, \dots, n_j-1, \dots, n_n}, \quad j = 1, \dots, n, \quad (36)$$

which, when iterated, lead to

$$c_{n_1, \dots, n_j, \dots, n_n} = \frac{z_j^{n_j}}{\sqrt{n_j!}} c_{n_1, \dots, 0, \dots, n_n}, \quad j = 1, \dots, n. \quad (37)$$

Hence, it turns out that

$$c_{n_1, \dots, n_n} = \frac{z_1^{n_1} \dots z_n^{n_n}}{\sqrt{n_1! \dots n_n!}} c_{0, \dots, 0}, \quad (38)$$

where $c_{0, \dots, 0}$ is to be found from the normalization condition. Thus the normalized AOCS become finally:

$$|z_1, \dots, z_n\rangle = \exp\left(-\frac{1}{2} \sum_{j=1}^n |z_j|^2\right) \sum_{n_1, \dots, n_n=0}^{\infty} \frac{z_1^{n_1} \dots z_n^{n_n}}{\sqrt{n_1! \dots n_n!}} |n_1, \dots, n_n\rangle, \quad (39)$$

up to a global phase factor.

3.2 Displacement Operator Coherent States (DOCS)

The displacement operator for the j -th oscillator mode of the Hamiltonian reads

$$D_j(z_j) = \exp(z_j B_j^\dagger - z_j^* B_j). \quad (40)$$

By using the BCH formula it turns out that

$$D_j(z_j) = \exp\left(-\frac{|z_j|^2}{2}\right) \exp(z_j B_j^\dagger) \exp(-z_j^* B_j). \quad (41)$$

Now, the global displacement operator is given by:

$$D(\mathbf{z}) \equiv D(z_1, \dots, z_n) = D_1(z_1) \cdots D_n(z_n), \quad (42)$$

where \mathbf{z} denotes the complex variables z_1, \dots, z_n associated to the n oscillator modes.

Let us obtain now the displacement operator coherent states (DOCS) $|\mathbf{z}\rangle$ from applying $D(\mathbf{z})$ onto the extremal state $|0, \dots, 0\rangle \equiv |\mathbf{0}\rangle$:

$$|\mathbf{z}\rangle = D(\mathbf{z})|\mathbf{0}\rangle = \exp\left(-\frac{1}{2} \sum_{j=1}^n |z_j|^2\right) \sum_{n_1, \dots, n_n=0}^{\infty} \frac{z_1^{n_1} \cdots z_n^{n_n} |n_1, \dots, n_n\rangle}{\sqrt{n_1! \cdots n_n!}}. \quad (43)$$

Notice that the AOCS and the DOCS are the same (compare Eqs.(39) and (43)).

3.3 Coherent state wave functions

In order to find the wave functions of the coherent states previously derived, we employ that $[z_j B_j^\dagger - z_j^* B_j, z_k B_k^\dagger - z_k^* B_k] = 0 \forall j, k$. Thus:

$$\begin{aligned} D(\mathbf{z}) &= \exp(z_1 B_1^\dagger - z_1^* B_1) \cdots \exp(z_n B_n^\dagger - z_n^* B_n) \\ &= \exp[(z_1 B_1^\dagger + \cdots + z_n B_n^\dagger) - (z_1^* B_1 + \cdots + z_n^* B_n)]. \end{aligned} \quad (44)$$

Using now Eq.(33) we can write

$$D(\mathbf{z}) = e^{-i(\vec{\Gamma} \cdot \vec{P} - \vec{\Sigma} \cdot \vec{X})} = e^{-\frac{i}{2} \vec{\Gamma} \cdot \vec{\Sigma}} e^{i \vec{\Sigma} \cdot \vec{X}} e^{-i \vec{\Gamma} \cdot \vec{P}} = e^{\frac{i}{2} \vec{\Gamma} \cdot \vec{\Sigma}} e^{-i \vec{\Gamma} \cdot \vec{P}} e^{i \vec{\Sigma} \cdot \vec{X}}, \quad (45)$$

where we have employed once again the BCH formula and we have taken

$$\vec{\Gamma} = 2\text{Re}[z_1^* \vec{\alpha}_1 + \cdots + z_n^* \vec{\alpha}_n], \quad \vec{\Sigma} = -2\text{Im}[z_1^* \vec{\beta}_1 + \cdots + z_n^* \vec{\beta}_n]. \quad (46)$$

Now, it is straightforward to find the wave function for the coherent state $|\mathbf{z}\rangle$,

$$\phi_{\mathbf{z}}(\vec{x}) = \langle \vec{x} | \mathbf{z} \rangle = \langle \vec{x} | D(\mathbf{z}) | \mathbf{0} \rangle = e^{-\frac{i}{2} \vec{\Gamma} \cdot \vec{\Sigma}} e^{i \vec{\Sigma} \cdot \vec{x}} \langle \vec{x} | e^{-i \vec{P} \cdot \vec{\Gamma}} | \mathbf{0} \rangle. \quad (47)$$

Since the operator \vec{P} is the coordinate displacement generator [1], it turns out that

$$\langle \vec{x} | e^{-i \vec{P} \cdot \vec{\Gamma}} = \langle \vec{x} - \vec{\Gamma} |, \quad (48)$$

so that

$$\phi_{\mathbf{z}}(\vec{x}) = e^{-\frac{i}{2}\vec{\Gamma}\cdot\vec{\Sigma}} e^{i\vec{\Sigma}\cdot\vec{x}} \langle \vec{x} - \vec{\Gamma} | 0 \rangle = e^{-\frac{i}{2}\vec{\Gamma}\cdot\vec{\Sigma}} e^{i\vec{\Sigma}\cdot\vec{x}} \phi_0(\vec{x} - \vec{\Gamma}). \quad (49)$$

A further calculation, using Eq.(31), leads finally to

$$\phi_{\mathbf{z}}(\vec{x}) = e^{-\frac{1}{2}(\vec{\Gamma}^T \mathbf{a} + i\vec{\Sigma})\cdot\vec{\Gamma}} e^{(\vec{\Gamma}^T \mathbf{a} + i\vec{\Sigma})\cdot\vec{x}} \phi_0(\vec{x}). \quad (50)$$

Once again, it becomes evident that the extremal state is important in our treatment, since its wave function determines the corresponding wave function for any other CS. Moreover, as it can be seen from Eq.(49), the position probability density for the CS $|\mathbf{z}\rangle$ becomes just a displaced version of the corresponding one for the extremal state $|0\rangle$.

4 Mathematical and physical properties

Let us derive next the completeness relationship for the previously derived coherent states. Notice that, from the point of view of the analysis of states in the Hilbert space of the system, this is the most important property which our CS would have [9, 11, 29, 30]. This is the reason why several authors use it as the fourth coherent state definition, considering it as the fundamental one which will survive in time (see e.g. [11]). We are going to calculate as well some important physical quantities in these states.

4.1 Completeness relationship

A straightforward calculation leads to:

$$\begin{aligned} & \left(\frac{1}{\pi}\right)^n \int \cdots \int |\mathbf{z}\rangle \langle \mathbf{z}| d^2 z_1 \dots d^2 z_n \\ &= \sum_{m_1, n_1, \dots, m_n, n_n=0}^{\infty} \frac{|m_1, \dots, m_n\rangle \langle n_1, \dots, n_n|}{\sqrt{m_1! n_1! \cdots m_n! n_n!}} \prod_{j=1}^n \left(\frac{1}{\pi} \int z_j^{m_j} z_j^{*n_j} e^{-|z_j|^2} d^2 z_j \right) = \mathbf{1}, \end{aligned} \quad (51)$$

with $\mathbf{1}$ being the identity operator. Thus, the coherent states $\{|\mathbf{z}\rangle\}$ form a complete set in the state space of the system (indeed they constitute an overcomplete set [31, 32]). This implies that any state can be expressed in terms of our coherent states, in particular, an arbitrary coherent state,

$$|\mathbf{z}'\rangle = \left(\frac{1}{\pi}\right)^n \int \cdots \int |\mathbf{z}\rangle \langle \mathbf{z}|\mathbf{z}'\rangle d^2 z_1 \dots d^2 z_n, \quad (52)$$

where the reproducing kernel $\langle \mathbf{z}|\mathbf{z}'\rangle$ is given by

$$\langle \mathbf{z}|\mathbf{z}'\rangle = \exp \left[-\frac{1}{2} \sum_{j=1}^n (|z_j|^2 - 2z_j^* z'_j + |z'_j|^2) \right]. \quad (53)$$

This means that, in general, our coherent states are not orthogonal to each other. Indeed, notice that inside our infinite set of coherent states only the extremal state of the system, $|0\rangle \equiv |\mathbf{z} = 0\rangle = |0, \dots, 0\rangle$, is also an eigenstate of the Hamiltonian.

4.2 Mean values of some physical quantities in a CS

Now we can calculate easily the mean values $\langle X_j \rangle_{\mathbf{z}} \equiv \langle \mathbf{z} | X_j | \mathbf{z} \rangle$, $\langle P_j \rangle_{\mathbf{z}} \equiv \langle \mathbf{z} | P_j | \mathbf{z} \rangle$, $j = 1, \dots, n$, in a given coherent state $|\mathbf{z}\rangle = |z_1, \dots, z_n\rangle$, as well as its mean square deviation in terms of the corresponding results for the extremal state $|0\rangle$. To do that, let us analyze first how the operators X_j , X_j^2 , P_j , P_j^2 are transformed under $D(\mathbf{z})$. By using Eqs.(45,46) it is straightforward to show that:

$$D^\dagger(\mathbf{z})X_j^n D(\mathbf{z}) = (X_j + \Gamma_j)^n, \quad D^\dagger(\mathbf{z})P_j^n D(\mathbf{z}) = (P_j + \Sigma_j)^n, \quad n = 1, 2, \dots \quad (54)$$

where we have used that, for an operator A which commutes with $[A, B]$, it turns out that

$$e^A B e^{-A} = B + [A, B] \Rightarrow e^A B^n e^{-A} = (B + [A, B])^n, \quad n = 1, 2, \dots \quad (55)$$

Thus, a straightforward calculation leads to

$$\langle X_j \rangle_{\mathbf{z}} \equiv \langle \mathbf{z} | X_j | \mathbf{z} \rangle = \langle 0 | D^\dagger(\mathbf{z}) X_j D(\mathbf{z}) | 0 \rangle = \langle X_j \rangle_0 + \Gamma_j. \quad (56)$$

On the other hand,

$$\langle X_j^2 \rangle_{\mathbf{z}} = \langle X_j^2 \rangle_0 + 2\Gamma_j \langle X_j \rangle_0 + \Gamma_j^2. \quad (57)$$

Hence,

$$(\Delta X_j)_{\mathbf{z}}^2 = \langle X_j^2 \rangle_{\mathbf{z}} - \langle X_j \rangle_{\mathbf{z}}^2 = \langle X_j^2 \rangle_0 - \langle X_j \rangle_0^2 = (\Delta X_j)_0^2. \quad (58)$$

Working in a similar way for P_j , it is obtained

$$\langle P_j \rangle_{\mathbf{z}} = \langle P_j \rangle_0 + \Sigma_j, \quad \langle P_j^2 \rangle_{\mathbf{z}} = \langle P_j^2 \rangle_0 + 2\Sigma_j \langle P_j \rangle_0 + \Sigma_j^2. \quad (59)$$

Then we have as well that

$$(\Delta P_j)_{\mathbf{z}}^2 = (\Delta P_j)_0^2, \quad (60)$$

i.e., the mean square deviations of X_j and P_j in the CS $|\mathbf{z}\rangle$ are independent of \mathbf{z} .

In order to end up this calculation, the mean values $\langle X_j \rangle_0$, $\langle P_j \rangle_0$, $\langle X_j^2 \rangle_0$, and $\langle P_j^2 \rangle_0$ for $j = 1, \dots, n$ are required. Let us describe now the procedure to find these $4n$ quantities. The first $2n$, $\langle X_j \rangle_0$, $\langle P_j \rangle_0$, can be easily found by recalling the definitions of B_k , B_k^\dagger (see section 2.1) and using the fact that their mean values in the extremal state $|0\rangle$ always vanish for $k = 1, \dots, n$:

$$\langle B_k \rangle_0 = \langle B_k^\dagger \rangle_0 = 0, \quad k = 1, \dots, n. \quad (61)$$

This is equivalent to the following linear system of $2n$ homogeneous equations

$$f_k^- \langle \eta \rangle_0 = f_k^+ \langle \eta \rangle_0 = 0, \quad k = 1, \dots, n. \quad (62)$$

Since the left eigenvectors f_k^\pm , $k = 1, \dots, n$, are linearly independent, the only solution for the $2n$ unknowns $\langle \eta \rangle_0$ is the trivial one, i.e., $\langle X_j \rangle_0 = \langle P_j \rangle_0 = 0$. It is worth to point out that this result simplifies Eqs.(56,57,59).

On the other hand, the mean values of the quadratic operators X_i^2 , P_i^2 , $i = 1, \dots, n$, in the extremal state $|0\rangle$ can be obtained from evaluating the corresponding quantities for the several non-equivalent products of pairs of annihilation B_j and creation B_k^\dagger operators. It is important to mention that these products should have the appropriate order to use the fact that B_j annihilates $|0\rangle$ and B_k^\dagger annihilates $\langle 0|$ (if the product involves one B_j it should be placed to the right while

if it involves one B_k^\dagger it should be placed to the left). In general, we will get $n(2n + 1)$ non-equivalent products of pairs of operators $B_i, B_j^\dagger, i, j = 1, \dots, n$: $n(n + 1)/2$ products of kind $B_i B_j, j = 1, \dots, n, i \leq j$; $n(n + 1)/2$ products of kind $B_i^\dagger B_j^\dagger, j = 1, \dots, n, i \leq j$; n^2 products of kind $B_i^\dagger B_j, i, j = 1, \dots, n$. The mean values in the extremal state $|0\rangle$ will lead to an inhomogeneous systems of $n(2n + 1)$ equations, with the same number of unknowns. When solving this system we will get $\langle X_j^2 \rangle_0, \langle P_j^2 \rangle_0, j = 1, \dots, n$, and the mean value of any other product of two canonical operators X_i, P_j .

It is customary nowadays to group the mean values of the quadratic products of the operators X_i, P_j in the coherent state $|\mathbf{z}\rangle$ in a $2n \times 2n$ real symmetric matrix $\sigma(\mathbf{z})$, called covariance matrix, whose elements are given by (remember that $\eta = (X_1, \dots, X_n, P_1, \dots, P_n)^T$):

$$\sigma_{ij}(\mathbf{z}) = \frac{1}{2} \langle \eta_i \eta_j + \eta_j \eta_i \rangle_{\mathbf{z}} - \langle \eta_i \rangle_{\mathbf{z}} \langle \eta_j \rangle_{\mathbf{z}}, \quad i, j = 1, \dots, 2n. \quad (63)$$

A straightforward calculation leads to

$$\sigma_{ij}(\mathbf{z}) = \sigma_{ij}(\mathbf{0}) = \frac{1}{2} \langle \eta_i \eta_j + \eta_j \eta_i \rangle_0 \equiv \sigma_{ij}, \quad (64)$$

where we have used that $\langle \eta_i \rangle_0 = 0, i = 1, \dots, 2n$. The conclusion is that the covariance matrix in our coherent state $|\mathbf{z}\rangle$ is once again independent from \mathbf{z} and depends just of the extremal state $|0\rangle$. Notice that the number of independent matrix elements σ_{ij} ($n(2n + 1)$) coincides with the number of unknowns which are determined from the set of $n(2n + 1)$ independent equations associated to the mean values of the quadratic products of B_j^\dagger, B_k in the extremal state $|0\rangle$.

Once the covariance matrix is determined, the generalized uncertainty relation can be evaluated [33–35]

$$\sigma_{ii} \sigma_{n+i, n+i} - \sigma_{i, n+i}^2 \geq \frac{1}{4}, \quad i = 1, \dots, n, \quad (65)$$

which coincides with the Robertson-Schrödinger uncertainty relation (see e.g. [33]).

Let us end up this section by calculating the mean value of the Hamiltonian in a given coherent state $|\mathbf{z}\rangle$. Equation (28) leads to

$$\langle H \rangle_{\mathbf{z}} = \langle \mathbf{z} | H | \mathbf{z} \rangle = \sum_{k=1}^n \gamma_k \omega_k |z_k|^2 + g'_0. \quad (66)$$

In order to get $\langle H^2 \rangle_{\mathbf{z}}$, let us notice that

$$H^2 = \sum_{j,k=1}^n \gamma_j \gamma_k \omega_j \omega_k B_j^\dagger B_k^\dagger B_j B_k + \sum_{k=1}^n \omega_k^2 B_k^\dagger B_k + 2g'_0 \sum_{k=1}^n \gamma_k \omega_k B_k^\dagger B_k + g_0'^2. \quad (67)$$

Thus we get

$$\langle H^2 \rangle_{\mathbf{z}} = \sum_{j,k=1}^n \gamma_j \gamma_k \omega_j \omega_k |z_j|^2 |z_k|^2 + \sum_{k=1}^n \omega_k^2 |z_k|^2 + 2g'_0 \sum_{k=1}^n \gamma_k \omega_k |z_k|^2 + g_0'^2. \quad (68)$$

Hence,

$$(\Delta H)_{\mathbf{z}}^2 = \sum_{k=1}^n \omega_k^2 |z_k|^2. \quad (69)$$

Notice that, for one-dimensional systems ($n = 1$), this expression reduces to the standard one for the harmonic oscillator (see e.g. [1]).

4.3 Time evolution of the CS

Suppose that at $t = 0$ our system is in a coherent state $|\mathbf{z}\rangle$. Thus, at a later time $t > 0$ the evolved state is found by acting on $|\mathbf{z}\rangle$ with the evolution operator of the system $U(t) = \exp(-iHt)$. By making use of Eq.(29) it turns out that

$$U(t) = e^{-ig'_0 t} \prod_{k=1}^n e^{-i\gamma_k \omega_k N_k}.$$

Hence,

$$U(t)|\mathbf{z}\rangle = e^{-ig'_0 t} |z_1(t), \dots, z_n(t)\rangle = e^{-ig'_0 t} |\mathbf{z}(t)\rangle, \quad (70)$$

where $z_j(t) = e^{-i\gamma_j \omega_j t} z_j = |z_j| e^{i(\theta_j - \gamma_j \omega_j t)}$. Equation (70) implies that a coherent state $|\mathbf{z}\rangle$ evolves in time into a new coherent state $|\mathbf{z}(t)\rangle = |z_1(t), \dots, z_n(t)\rangle$, where the j -th degree of freedom $z_j(t)$ just rotates at its characteristic frequency ω_j (clockwise if $\gamma_j = 1$ and counter-clockwise if $\gamma_j = -1$).

4.4 Gazeau-Klauder coherent states

At this point, it would be interesting to check if our CS belong to the class introduced recently by Gazeau and Klauder [36]. Using their notation, for a system with a Hamiltonian \mathcal{H} such that the ground state energy is zero, the Gazeau-Klauder CS $\{|J, \theta\rangle, J \geq 0, -\infty < \theta < \infty\}$ obey the following properties:

- (a) Continuity: $(J', \theta') \rightarrow (J, \theta) \Rightarrow |J', \theta'\rangle \rightarrow |J, \theta\rangle$.
- (b) Resolution of unity: $\mathbf{1} = \int |J, \theta\rangle \langle J, \theta| d\mu(J, \theta)$.
- (c) Temporal stability: $e^{-i\mathcal{H}t} |J, \theta\rangle = |J, \theta + \omega t\rangle, \omega = \text{constant}$.
- (d) Action identity: $\langle J, \theta | \mathcal{H} | J, \theta \rangle = \omega J$.

Concerning the first property, it is straightforward to check that our coherent states given in Eq.(43) are such that $|\mathbf{z}'\rangle \rightarrow |\mathbf{z}\rangle$ as $\mathbf{z}' \rightarrow \mathbf{z}$, i.e., they are continuous in \mathbf{z} . As for the second and third properties, both were explicitly proven in sections 4.1 and 4.3 respectively. It remains just to analyze if it is valid the action identity given in (d). Let us notice first of all that it is valid for each partial Hamiltonian $H_k = \gamma_k \omega_k N_k$ of our system,

$$\langle \mathbf{z} | H_k | \mathbf{z} \rangle = \gamma_k \omega_k |z_k|^2,$$

which is time-independent. Therefore, property (d) becomes valid for each degree of freedom separately and thus it is valid for our global system with the natural identification $J_k = |z_k|^2$, $\theta_k = \arg(z_k)$ so that

$$\langle \mathbf{z} | (H - g'_0) | \mathbf{z} \rangle = \sum_{k=1}^n \gamma_k \omega_k J_k.$$

We conclude that our CS of Eq.(43) become as well an n -dimensional generalization of the Gazeau-Klauder CS if we express each complex component z_k of \mathbf{z} in its polar form (the polar coordinates essentially coincide with the canonical action-angle variables for the corresponding classical system).

5 Asymmetric Penning trap coherent states

Let us apply now the previous technique to the asymmetric Penning trap. Such an arrangement can be used to control some quantum mechanical phenomena [37] as well as to perform high-precision measurements of fundamental properties of particles. Moreover, it is a quite natural system to analyze the decoherence taking place due to the unavoidable interaction of the system with its environment [38, 39]. Since the asymmetric Penning trap becomes the ideal one when the asymmetry parameter vanishes [40–42], it will be straightforward to compare these results with those recently obtained for the ideal Penning trap [16] (see also [35, 43, 44]).

The Hamiltonian of a charged particle with mass m and charge q in an asymmetric Penning trap reads

$$H = \frac{\vec{P}^2}{2m} + \frac{\omega_c}{2}(XP_y - YP_x) + \frac{m}{2}(\omega_x^2 X^2 + \omega_y^2 Y^2 + \omega_z^2 Z^2), \quad (71)$$

$\omega_c = qB/m$ and ω_z being the cyclotron and axial frequencies respectively, and the frequencies ω_x, ω_y are given by

$$\omega_x^2 = \frac{\omega_c^2}{4} - \frac{\omega_z^2}{2}(1 + \varepsilon), \quad \omega_y^2 = \frac{\omega_c^2}{4} - \frac{\omega_z^2}{2}(1 - \varepsilon), \quad (72)$$

where $|\varepsilon| < 1$ is the real asymmetry parameter and we are denoting $\vec{P} = (P_x, P_y, P_z)^T$, $\vec{X} = (X, Y, Z)^T$. Without loosing generality [16], from now on we will assume that $m = 1$.

As it was seen at section 2, the main role in our treatment is played by the matrix Λ such that $[iH, \eta] = \Lambda\eta$. We choose here $\eta = (X, Y, P_x, P_y, Z, P_z)^T$ so that

$$\Lambda = \begin{pmatrix} 0 & -\omega_c/2 & 1 & 0 & 0 & 0 \\ \omega_c/2 & 0 & 0 & 1 & 0 & 0 \\ -\omega_x^2 & 0 & 0 & -\omega_c/2 & 0 & 0 \\ 0 & -\omega_y^2 & \omega_c/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & 0 \end{pmatrix}. \quad (73)$$

The eigenvalues (λ) of Λ are

$$\begin{aligned} \lambda_1 &= \frac{i\omega_c}{2}\sqrt{2 - \delta + R} = i\omega_1, & \lambda_2 &= \frac{i\omega_c}{2}\sqrt{2 - \delta - R} = i\omega_2, \\ \lambda_3 &= i\omega_z = i\omega_3, & R &= \sqrt{4(1 - \delta) + \delta^2\varepsilon^2}, & 0 < \delta &= \frac{2\omega_z^2}{\omega_c^2} < 1, \end{aligned} \quad (74)$$

and their corresponding complex conjugate. The right (u) and left (f) eigenvectors of Λ become

$$\begin{aligned} u_1^+ &= s_1 \left(\frac{4}{\omega_c(\delta\varepsilon + R)}, -\frac{i}{\omega_1} \frac{2 + \delta\varepsilon + R}{\delta\varepsilon + R}, \frac{i\omega_c}{2\omega_1} \frac{2(1 - \delta) - \delta\varepsilon + R}{\delta\varepsilon + R}, 1, 0, 0 \right)^T, \\ u_2^+ &= s_2 \left(\frac{4}{\omega_c(\delta\varepsilon - R)}, -\frac{i}{\omega_2} \frac{2 + \delta\varepsilon - R}{\delta\varepsilon - R}, \frac{i\omega_c}{2\omega_2} \frac{2(1 - \delta) - \delta\varepsilon - R}{\delta\varepsilon - R}, 1, 0, 0 \right)^T, \\ u_3^+ &= s_3 \left(0, 0, 0, 0, -\frac{i}{\omega_z}, 1 \right)^T, \end{aligned}$$

$$\begin{aligned}
f_1^+ &= t_1 \left(\frac{\omega_c}{4}(R - \delta\varepsilon), \frac{i\omega_c^2}{8\omega_1}[2(1 - \delta) + \delta\varepsilon + R], -\frac{i\omega_c}{4\omega_1}(2 - \delta\varepsilon + R), 1, 0, 0 \right), \\
f_2^+ &= t_2 \left(\frac{\omega_c}{4}(-R - \delta\varepsilon), \frac{i\omega_c^2}{8\omega_2}[2(1 - \delta) + \delta\varepsilon - R], -\frac{i\omega_c}{4\omega_2}(2 - \delta\varepsilon - R), 1, 0, 0 \right), \\
f_3^+ &= t_3 (0, 0, 0, 0, i\omega_z, 1),
\end{aligned} \tag{75}$$

where $s_j, t_j \in \mathbb{C}, j = 1, 2, 3$. The requirement that the right and left eigenvectors be dual to each other implies

$$s_1 = \frac{1}{4t_1} \left(1 + \frac{\delta\varepsilon}{R} \right), \quad s_2 = \frac{1}{4t_2} \left(1 - \frac{\delta\varepsilon}{R} \right), \quad s_3 = \frac{1}{2t_3}. \tag{76}$$

On the other hand, up to some phase factors, the condition imposed by Eqs.(10,11) leads to,

$$t_1 = \frac{1}{\sqrt{2i(f_{1a}f_{1c} - f_{1b})}}, \quad t_2 = \frac{1}{\sqrt{2i(f_{2b} - f_{2a}f_{2c})}}, \quad t_3 = \frac{1}{\sqrt{2\omega_3}}, \tag{77}$$

where we are denoting $f_1^+ = t_1(f_{1a}, f_{1b}, f_{1c}, 1, 0, 0)$, $f_2^+ = t_2(f_{2a}, f_{2b}, f_{2c}, 1, 0, 0)$ in order to simplify the notation (compare Eq.(75)). Moreover, the crucial signs for us to conclude that our asymmetric Penning trap Hamiltonian is not positively defined become:

$$\gamma_1 = 1, \quad \gamma_2 = -1, \quad \gamma_3 = 1. \tag{78}$$

Thus, our annihilation operators take the form (see Eq.(22)):

$$\begin{aligned}
B_1 = L_1^- &= t_1 (f_{1a}X - f_{1b}Y - f_{1c}P_x + P_y), \\
B_2 = L_2^+ &= t_2 (f_{2a}X + f_{2b}Y + f_{2c}P_x + P_y), \\
B_3 = L_3^- &= t_3 (-i\omega_3 Z + P_z).
\end{aligned} \tag{79}$$

From these operators and their hermitian conjugates, it is straightforward to identify the α_j and β_j , $j = 1, 2, 3$ which allow us to find the matrix \mathbf{a} such that $\mathbf{a}\alpha_j = \beta_j$. Its matrix elements a_{ij} become now:

$$\begin{aligned}
a_{11} &= -i \frac{f_{1a} - f_{2a}}{f_{1c} + f_{2c}}, \quad a_{12} = a_{21} = i \frac{f_{1b} + f_{2b}}{f_{1c} + f_{2c}}, \quad a_{22} = i \frac{f_{1c}f_{2b} - f_{2c}f_{1b}}{f_{1c} + f_{2c}}, \\
a_{33} &= \omega_3, \quad a_{13} = a_{31} = a_{23} = a_{32} = 0.
\end{aligned} \tag{80}$$

It can be shown that $a_{11}, a_{22}, a_{33} \in \mathbb{R}^+$ while a_{12} is purely imaginary. Thus the extremal state wave function of Eq.(31) acquires the form:

$$\phi_0(\vec{x}) = c \exp \left(-\frac{1}{2}a_{11}x^2 - \frac{1}{2}a_{22}y^2 - a_{12}xy \right) \exp \left(-\frac{1}{2}a_{33}z^2 \right). \tag{81}$$

The associated eigenvalue becomes $E_{0,0,0} = (\omega_1 - \omega_2 + \omega_3)/2$.

Concerning the coherent states $|z_1, z_2, z_3\rangle$, the general treatment developed in section 3 is straightforwardly applicable, and their explicit expressions are given by Eq.(43) with $n = 3$. Their corresponding wave functions are given by

$$\phi_z(\vec{x}) = \langle \vec{x} | \mathbf{z} \rangle = e^{-i\vec{\Gamma} \cdot \vec{\Sigma}/2} e^{i\vec{\Sigma} \cdot \vec{x}} \phi_0(x - \Gamma_1, y - \Gamma_2, z - \Gamma_3), \tag{82}$$

where

$$\vec{\Gamma} = 2 \begin{pmatrix} it_1 f_{1c} \text{Re}[z_1] - it_2 f_{2c} \text{Re}[z_2] \\ -t_1 \text{Im}[z_1] - t_2 \text{Im}[z_2] \\ -t_3 \text{Im}[z_3] \end{pmatrix}, \quad \vec{\Sigma} = 2 \begin{pmatrix} t_1 f_{1a} \text{Im}[z_1] + t_2 f_{2a} \text{Im}[z_2] \\ -it_1 f_{1b} \text{Re}[z_1] + it_2 f_{2b} \text{Re}[z_2] \\ t_3 \omega_3 \text{Re}[z_3] \end{pmatrix}. \quad (83)$$

The mean values $\langle X_j \rangle_{\mathbf{z}}, \langle P_j \rangle_{\mathbf{z}}$, immediately follow from Eqs.(56, 59) with $\langle X_j \rangle_0 = \langle P_j \rangle_0 = 0$, i.e.,

$$\langle X_j \rangle_{\mathbf{z}} = \Gamma_j, \quad \langle P_j \rangle_{\mathbf{z}} = \Sigma_j, \quad j = 1, 2, 3. \quad (84)$$

As for the mean values of the quadratic operators in the extremal state, we have solved the system of equations arising from the null mean values of the products of pairs of annihilation B_j and creation B_k^\dagger operators. We get

$$\begin{aligned} \langle X^2 \rangle_0 &= \frac{1}{2a_{11}}, & \langle P_x^2 \rangle_0 &= \frac{1}{2} \left(a_{11} - \frac{a_{12}^2}{a_{22}} \right), \\ \langle Y^2 \rangle_0 &= \frac{1}{2a_{22}}, & \langle P_y^2 \rangle_0 &= \frac{1}{2} \left(a_{22} - \frac{a_{12}^2}{a_{11}} \right), \\ \langle Z^2 \rangle_0 &= \frac{1}{2a_{33}}, & \langle P_z^2 \rangle_0 &= \frac{1}{2} a_{33}, \end{aligned} \quad (85)$$

and the crossed products

$$\begin{aligned} \langle XP_x \rangle_0 &= \frac{i}{2}, & \langle XP_y \rangle_0 &= \frac{i a_{12}}{2 a_{11}}, & \langle XP_z \rangle_0 &= 0, \\ \langle YP_x \rangle_0 &= \frac{i a_{12}}{2 a_{22}}, & \langle YP_y \rangle_0 &= \frac{i}{2}, & \langle YP_z \rangle_0 &= 0, \\ \langle ZP_x \rangle_0 &= 0, & \langle ZP_y \rangle_0 &= 0, & \langle ZP_z \rangle_0 &= \frac{i}{2}. \end{aligned} \quad (86)$$

Therefore, using Eqs.(58, 60) we get the Heisenberg uncertainty relationships

$$\begin{aligned} (\Delta X)_{\mathbf{z}}^2 (\Delta P_x)_{\mathbf{z}}^2 &= (\Delta Y)_{\mathbf{z}}^2 (\Delta P_y)_{\mathbf{z}}^2 = \frac{1}{4} \left(1 + \frac{|a_{12}|^2}{a_{11} a_{22}} \right) \geq \frac{1}{4}, \\ (\Delta Z)_{\mathbf{z}}^2 (\Delta P_z)_{\mathbf{z}}^2 &= \frac{1}{4}, \end{aligned} \quad (87)$$

while Eq.(69) with $n = 3$ gives the mean square deviation for the Hamiltonian.

Once we have calculated the mean values of the quadratic products given in Eqs.(85,86), it is straightforward to evaluate the covariance matrix elements of Eq.(64). With the ordering $\eta = (X, Y, P_x, P_y, Z, P_z)^T$, it is obtained:

$$\sigma = \begin{pmatrix} (\Delta X)_0^2 & 0 & 0 & \frac{ia_{12}}{2a_{11}} & 0 & 0 \\ 0 & (\Delta Y)_0^2 & \frac{ia_{12}}{2a_{22}} & 0 & 0 & 0 \\ 0 & \frac{ia_{12}}{2a_{22}} & (\Delta P_x)_0^2 & 0 & 0 & 0 \\ \frac{ia_{12}}{2a_{11}} & 0 & 0 & (\Delta P_y)_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\Delta Z)_0^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\Delta P_z)_0^2 \end{pmatrix}. \quad (88)$$

Notice that this covariance matrix is non-diagonal. However, since $\sigma_{13} = \sigma_{24} = \sigma_{56} = 0$, it turns out that the generalized uncertainty relations of Eq.(65) reduce to the Heisenberg uncertainty relations given in Eq.(87).

A plot of $(\Delta X)_z(\Delta P_x)_z$ as a function of the parameters ε and δ is given in Fig.1. As it can be seen from Eqs.(80,87) and from Fig.1, the coherent states minimize the Heisenberg uncertainty relationship for $\varepsilon = 0$, which coincides with the results recently obtained for the ideal Penning trap [16]. However, for $\varepsilon \neq 0$ it turns out that $(\Delta X)_z(\Delta P_x)_z > 1/2$. Notice that the same plot will appear for the uncertainty product $(\Delta Y)_z(\Delta P_y)_z$.

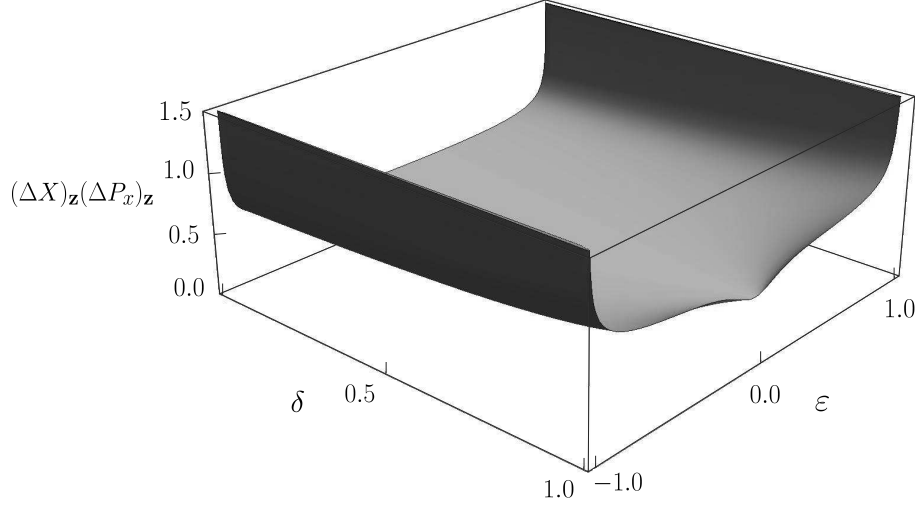


Figure 1: Heisenberg uncertainty relationship $(\Delta X)_z(\Delta P_x)_z$ for the asymmetric Penning trap coherent states as function of the real dimensionless parameters $|\varepsilon| < 1$, $0 < \delta < 1$.

6 Concluding remarks

In this work we have proposed a systematic technique to find the CS for systems governed by quadratic Hamiltonians in the trap regime. To do this, we introduced a prescription to identify in a simple way the appropriate ladder operators which play the same role as the annihilation and creation operators for the 1-dimensional harmonic oscillator. These operators allowed us to generate the eigenvectors and eigenvalues for the Hamiltonian departing from the extremal state, the analogue of the ground state although it is not necessarily an eigenstate associated to the lowest possible eigenvalue. The explicit expression for the extremal state wave function was as well explicitly calculated.

For systems governed by this kind of Hamiltonians the two algebraic CS definitions (either as simultaneous eigenstates of the annihilation operators or as resulting from the action of the displacement operator onto the extremal state) lead to the same set of states. The explicit expression for the corresponding wave functions has been also derived.

We have calculated explicitly the mean values of the position and momentum operators in an arbitrary coherent state. Moreover, we have provided as well a prescription to obtain

algebraically, by solving a linear systems of equations, the mean values of the quadratic products of these operators in the CS.

Through this method we have found the asymmetric Penning trap coherent states and we have explored some of their physical properties. In particular, it is worth to point out that, in general, they do not minimize the Heisenberg uncertainty relationship. The differences from the minimum are induced by the deviations of the axial symmetry which the ideal Penning trap has (measured by the asymmetry parameter ε).

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